ISOMETRIC DILATIONS AND VON NEUMANN INEQUALITY FOR A CLASS OF TUPLES IN THE POLYDISC

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Abstract. The celebrated Sz.-Nagy and Foias and Ando theorems state that a single contraction, or a pair of commuting contractions, acting on a Hilbert space always possesses isometric dilation and subsequently satisfies the von Neumann inequality for polynomials in $\mathbb{C}[z]$ or $\mathbb{C}[z_1, z_2]$, respectively. However, in general, neither the existence of isometric dilation nor the von Neumann inequality holds for n-tuples, $n \geq 3$, of commuting contractions. The goal of this paper is to provide a taste of isometric dilations, von Neumann inequality and a refined version of von Neumann inequality for a large class of n-tuples, $n \geq 3$, of commuting contractions.

1. Introduction

In this paper we investigate isometric dilation, von Neumann inequality and a refined version of von Neumann inequality, in terms of algebraic variety in the polydisc $\mathbb{D}^n$, for a large class of n-tuples, $n \geq 3$, of commuting contractions on Hilbert spaces. The set of all ordered n-tuples of commuting contractions on a Hilbert space $\mathcal{H}$ will be denoted as $\mathcal{T}^n(\mathcal{H})$, that is

$$\mathcal{T}^n(\mathcal{H}) = \{(T_1, \ldots, T_n) : T_i \in \mathcal{B}(\mathcal{H}), \|T_i\| \leq 1, T_iT_j = T_jT_i, 1 \leq i, j \leq n\},$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. Here we are mostly interested in n-tuples, $n \geq 3$, of commuting contractions as it is well-known that a contraction, or a pair of commuting contractions, admits isometric dilation and hence, satisfies the von Neumann inequality (see Sz.-Nagy and Foias [24] and Ando [4]). A refined version of von Neumann inequality, in the sense of algebraic varieties, also follows from the recent papers [3], [13] and [14]. More specifically, here we are concerned with the validity of the following three statements for tuples in $\mathcal{T}^n(\mathcal{H})$.

Statement 1 (On isometric dilations): *Let $T \in \mathcal{T}^n(\mathcal{H})$. Then there exist a Hilbert space $\mathcal{K}(\supseteq \mathcal{H})$ and an n-tuple of commuting isometries $V \in \mathcal{T}^n(\mathcal{K})$ such that T dilates to V.*

Now any n-tuple of commuting isometries $V \in \mathcal{T}^n(\mathcal{K})$ can be extended to an n-tuple of commuting unitaries, that is, there exist a Hilbert space $\mathcal{L}$ containing $\mathcal{K}$ and an n-tuple of commuting unitary operators $U \in \mathcal{T}^n(\mathcal{L})$ which extends V [24]. Therefore, the celebrated von Neumann inequality is an immediate consequence of Statement 1 (cf. [24]):

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Statement 2 (On von Neumann inequality): If $T \in \mathcal{T}^n(\mathcal{H})$, then for all $p \in \mathbb{C}[z_1, \ldots, z_n]$, the following holds:

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\boldsymbol{z} \in \mathbb{D}^n} |p(\boldsymbol{z})|.$$

Here $\boldsymbol{z}$ denotes the element $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$, $z_i \in \mathbb{C}$, and $\mathbb{D}^n = \{\boldsymbol{z} \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$.

The next natural geometric and algebraic question to consider, after Agler and McCarthy [3], is the existence of varieties in $\mathbb{D}^n$ in the von Neumann inequality:

Statement 3 (On a refined von Neumann inequality): Let $T \in \mathcal{T}^n(\mathcal{H})$. Then there exists an algebraic variety V, depending on T, in $\mathbb{D}^n$ (or in $\overline{\mathbb{D}^n}$) such that for all $p \in \mathbb{C}[z_1, \ldots, z_n]$, the following holds:

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\boldsymbol{z} \in V} |p(\boldsymbol{z})|.$$

As we hinted earlier, Statement 2, and hence Statement 1, fails spectacularly in the sense that the von Neumann inequality does not hold in general for $n \geq 3$. This result is due to Varopoulos [28] and Crabb and Davie [11]. On the other hand, by presenting a list of elementary counterexamples, Parrott [25] proved that triples of commuting contractions do not, in general, possess commuting isometric dilations. We refer the reader interested in deep subtleties of von Neumann inequality for n-tuples of commuting contractions to Choi and Davidson [9], Drury [16], Holbrook [19, 20], Knese [21], Kosiński [22] and Pisier [26].

We also point out here an important difference between the Sz.-Nagy and Foias dilation [24] for contractions and Ando dilation [4] for pairs of commuting contractions. In the former case, the dilating isometries are explicit in the sense of the classical Wold and von Neumann decomposition [24]. In the latter case, dilating pairs of commuting isometries are complicated and mostly unclassified. This leads us to a reformulation of Statement 1:

Statement 1* (On explicit isometric dilations): *Let $T \in \mathcal{T}^n(\mathcal{H})$. Then there exist a Hilbert space $\mathcal{K}(\supseteq \mathcal{H})$ and an n-tuple of explicit (or tractable) commuting isometries $V \in \mathcal{T}^n(\mathcal{K})$ such that T dilates to V.*

We refer the reader to [3], [13] and [14] for classes of pairs of commuting contractions with explicit (or tractable) dilating isometries.

The above discussion leads naturally to the question of determining n-tuples of operators in $\mathcal{T}^n(\mathcal{H})$, $n \geq 3$, satisfying Statements 1, 2, 3 and 1*. This research direction is still mostly unexplored except for the work of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman [18]. More specifically, and elegantly, Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman [18] proved the validity of Statement 2 for a large class of n-tuples of commuting strict contractions, $n \geq 3$. In other words, if an n-tuple, $n \geq 3$, of commuting strict contractions T obeys certain positivity condition, then the open unit polydisc is a spectral set for T. This also yields, following Arveson's notion of completely bounded maps (see [5], [6] and Corollary 4.9 in [26]), existence of unitary dilations for those n-tuples of commuting strict contractions. The main stimulus for their work was provided by scattering theory, Schur-Agler class of functions and de Branges-Rovnyak models [15] in several variables. This is also the spirit behind results by Cotlar and Sadosky [10], Agler and McCarthy [1], Eschmeier and Putinar [17] and many more.

In this paper, we introduce a large class, namely $\mathcal{T}^n_{p,q}(\mathcal{H})$ (see Subsection 2.3), of n-tuples, $n \geq 3$, of commuting contractions and show that they dilate to n-tuples of explicit commuting isometries. Therefore, Statement 1* and hence Statement 1 holds for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$. This also allows us to prove the von Neumann inequality for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$ (that is, Statement 2 holds). In particular, in a larger context (see the examples in Subsection 2.3), we prove that the Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman's n-tuples of operators [18] admit explicit isometric dilations and hence yield the von Neumann inequality. Our recipe even provides sharper results with new proofs of the results of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman. Here, however, our treatment of dilations and von Neumann inequality is conceptually different. Our von Neumann inequality is even stronger for finite rank n-tuples of operators in the sense of algebraic varieties (and so, Statement 3 holds). Furthermore, our technique offers some geometric, analytic and algebraic structural insight into the positivity assumptions of n-tuples of operators. Our methodology is motivated by the Hilbert module approach to multivariable operator theory (cf. [27]).

The rest of the paper is organized as follows. Section 2 introduces terminology used throughout this paper. This section also gives a list of motivating and non-trivial examples of tuples of commuting contractions. Section 3 establishes the existence, with explicit constructions, of isometric dilations for a large class of finite rank n-tuples of commuting contractions. Using the isometric dilations, in Section 4, we obtain a refined version of von Neumann inequality (in terms of an algebraic variety) for finite rank n-tuples of commuting contractions. Finally, in Section 5 we consider the more general problem of describing isometric dilations for n-tuples of commuting contractions. Sections 3 and 5 are independent of each other.

2. Definitions and Examples

This section is aimed at providing definitions, motivating examples and a known dilation theorem on n-tuples of commuting contractions. First, we introduce some standard notation that will be used in this paper. We denote

$$\mathbb{Z}^n_+ = \{\boldsymbol{k} = (k_1, \dots, k_n) : k_i \in \mathbb{Z}_+, i = 1, \dots, n\}.$$

Also for each multi-index $\boldsymbol{k} \in \mathbb{Z}^n_+$, commuting tuple $T = (T_1, \dots, T_n)$ on a Hilbert space $\mathcal{H}$, and $\boldsymbol{z} \in \mathbb{C}^n$ we denote

$$T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n},$$

and

$$\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}.$$

We begin with the definition of isometric dilations for n-tuples of commuting contractions.

2.1. **Dilations of commuting tuples.** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $T \in \mathcal{T}^n(\mathcal{H})$ and $V \in \mathcal{T}^n(\mathcal{K})$. Then V is said to be an *isometric dilation* of T if V is an n-tuple of commuting isometries and there exists an isometry $\Pi : \mathcal{H} \to \mathcal{K}$ such that $\Pi T_i^* = V_i^* \Pi$ for all $i = 1, \dots, n$. We also say that T *dilates* to V.

In this case, for $\boldsymbol{k} \in \mathbb{Z}^n_+$, we have

$$\Pi T^{*\boldsymbol{k}} = V^{*\boldsymbol{k}} \Pi,$$

and so

$$\Pi T^{*\boldsymbol{k}}\Pi^* = V^{*\boldsymbol{k}}\Pi\Pi^*,$$

since

$$(\Pi\Pi^*)V^{*\boldsymbol{k}}(\Pi\Pi^*) = V^{*\boldsymbol{k}}(\Pi\Pi^*),$$

or, equivalently $V^{*\boldsymbol{k}}\mathcal{Q} \subseteq \mathcal{Q}$, where

$$\mathcal{Q} = ran\,\Pi.$$

This immediately yields the following: $(T_1, \ldots, T_n)$ on $\mathcal{H}$ and $(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}})$ on $\mathcal{Q}$ are unitarily equivalent under the isometric isomorphism $\Pi : \mathcal{H} \to \mathcal{Q}$, and

$$(P_{\mathcal{Q}}V|_{\mathcal{Q}})^{*\boldsymbol{k}} = V^{*\boldsymbol{k}}|_{\mathcal{Q}},$$

for all $\boldsymbol{k} \in \mathbb{Z}_+^n$. Here $P_{\mathcal{Q}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{Q}$. Therefore, the n-tuple T has a power dilation to the n-tuple of commuting isometries V, in the classical sense of Sz.-Nagy and Foias and Halmos.

The following example of isometric dilation is typical: Let $H^2(\mathbb{D}^n)$, the Hardy space over $\mathbb{D}^n$, be the space of all analytic functions $f = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} a_{\boldsymbol{k}}\boldsymbol{z}^{\boldsymbol{k}}$ on $\mathbb{D}^n$ for which the norm

$$\|f\|_{H^2(\mathbb{D}^n)} = (\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} |a_{\boldsymbol{k}}|^2)^{\frac{1}{2}} < \infty.$$

Let $(M_{z_1}, \ldots, M_{z_n})$ denote the n-tuple of multiplication operators on $H^2(\mathbb{D}^n)$ defined by

$$(M_{z_i}f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}),$$

for all $f \in H^2(\mathbb{D}^n)$, $\boldsymbol{w} \in \mathbb{D}^n$ and $i = 1, \ldots, n$. Then $(M_{z_1}, \ldots, M_{z_n})$ is an n-tuple of commuting isometries $(M_{z_1}, \ldots, M_{z_n})$ on the Hardy space $H^2(\mathbb{D}^n)$. Now for a joint $(M_{z_1}^*, \ldots, M_{z_n}^*)$-invariant subspace $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$, consider

$$T_j = P_{\mathcal{Q}}M_{z_j}|_{\mathcal{Q}},$$

and

$$\Pi = i,$$

where $i : \mathcal{Q} \hookrightarrow H^2(\mathbb{D}^n)$ is the natural inclusion map. Then

$$\Pi T_j^* = M_{z_j}^*\Pi,$$

for all $j = 1, \ldots, n$. This implies that $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$ is an isometric dilation of $(T_1, \ldots, T_n)$ on $\mathcal{Q}$.

2.2. **Hardy space and dilations.** We denote by $\mathbb{S}_n$ the Szegö kernel on $\mathbb{D}^n$, that is,

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n (1 - z_i\bar{w}_i)^{-1},$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$. Then $H^2(\mathbb{D}^n)$ is known to be a reproducing kernel Hilbert space with kernel $\mathbb{S}_n$. If $\mathcal{E}$ is a Hilbert space, then $H^2_{\mathcal{E}}(\mathbb{D}^n)$ denotes the $\mathcal{E}$-valued Hardy space over $\mathbb{D}^n$. Also as usual, $H^2_{\mathcal{E}}(\mathbb{D}^n)$ will be identified with the Hilbert space tensor product $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ via the natural unitary map $\boldsymbol{z}^{\boldsymbol{k}}\eta \mapsto \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta$ for all $\boldsymbol{k} \in \mathbb{Z}_+^n$ and $\eta \in \mathcal{E}$. It is a well-known fact that

$H^2_{\mathcal{E}}(\mathbb{D}^n)$ is a reproducing kernel Hilbert space on $\mathbb{D}^n$ corresponding to the $\mathcal{B}(\mathcal{E})$-valued kernel function

$$(\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}}.$$

The n-tuple of multiplication operators $(M_{z_1}, \dots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ defined analogously by

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}),$$

for all $f \in H^2_{\mathcal{E}}(\mathbb{D}^n)$, $\boldsymbol{w} \in \mathbb{D}^n$ and $i = 1, \dots, n$. Let $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^n)$ denote the set of all bounded $\mathcal{B}(\mathcal{E})$-valued analytic functions on $\mathbb{D}^n$. The following is a well-known fact (cf. page 655 in [7]): If $X \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^n))$, then $X M_{z_i} = M_{z_i} X$ if and only if $X = M_\Theta$ for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^n)$. Now note that

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k} \in \{0,1\}^n} (-1)^{|\boldsymbol{k}|} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{k}},$$

where $|\boldsymbol{k}| = \sum_i k_i$, $\boldsymbol{k} \in \mathbb{Z}^n_+$. With this motivation, for every $T \in \mathcal{T}^n(\mathcal{H})$ we set

$$\mathbb{S}_n^{-1}(T, T^*) = \sum_{\boldsymbol{k} \in \{0,1\}^n} (-1)^{|\boldsymbol{k}|} T^{\boldsymbol{k}} T^{*\boldsymbol{k}}.$$

The set of all $T \in \mathcal{T}^n(\mathcal{H})$ with $\mathbb{S}_n^{-1}(T, T^*) \geq 0$ will be denoted by $\mathbb{S}_n(\mathcal{H})$, that is

$$\mathbb{S}_n(\mathcal{H}) = \{T \in \mathcal{T}^n(\mathcal{H}) : \mathbb{S}_n^{-1}(T, T^*) \geq 0\}.$$

A tuple $T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H})$ is said to be *pure* if $\|T_i^{*m} h\| \to 0$ for all $h \in \mathcal{H}$ and $i = 1, \dots, n$.

The following theorem on pure n-tuples in $\mathbb{S}_n(\mathcal{H})$ is one of the most definite and significant results in multivariable dilation theory (see [12] and [23]).

Theorem 2.1. *Let $T \in \mathbb{S}_n(\mathcal{H})$ be a pure tuple. If*

$$D_T = \mathbb{S}_n^{-1}(T, T^*)^{1/2},$$

and

$$\mathcal{D}_T = \overline{ran}\, \mathbb{S}_n^{-1}(T, T^*),$$

then $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_T}(\mathbb{D}^n)$ defined by

$$(\Pi h)(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \boldsymbol{z}^{\boldsymbol{k}} D_T T^{*\boldsymbol{k}} h,$$

for all $\boldsymbol{z} \in \mathbb{D}^n$ and $h \in \mathcal{H}$, is an isometry and $\Pi T_i^* = M_{z_i}^* \Pi$ for all $i = 1, \dots, n$. In particular, T on $\mathcal{H}$ dilates to $(M_{z_1}, \dots, M_{z_n})$ on $H^2_{\mathcal{D}_T}(\mathbb{D}^n)$.

In the sequel, the isometry Π defined in the above theorem will be referred to as *canonical isometry* corresponding to T.

2.3. **Commuting tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.** We now introduce the central object of this paper.

Let $\mathcal{H}$ be a Hilbert space, and let $n \geq 3$ and $1 \leq p < q \leq n$ be fixed throughout the article. Let $T \in \mathcal{T}^n(\mathcal{H})$. For each $i \in \{1, \ldots, n\}$, we define

$$\hat{T}_i = (T_1, \ldots, T_{i-1}, T_{i+1}, T_{i+2}, \ldots, \ldots, T_n) \in \mathcal{T}^{(n-1)}(\mathcal{H}),$$

the $(n-1)$-tuple obtained from T by removing T_i. Define

$$\mathcal{T}^n_{p,q}(\mathcal{H}) = \{T \in \mathcal{T}^n(\mathcal{H}) : \hat{T}_p, \hat{T}_q \in \mathbb{S}_{n-1}(\mathcal{H}) \text{ and } \hat{T}_p \text{ is pure}\}.$$

For example, let $n = 3$, $p = 1$ and $q = 2$. Then $(T_1, T_2, T_3) \in \mathcal{T}^3_{1,2}(\mathcal{H})$ if and only if:
(i) $\|T_i\| \leq 1$ for all $i = 1, 2, 3$,
(ii) $\hat{T}_1 = (T_2, T_3)$ is pure (that is, $\|T_i^{*m} h\| \to 0$ as $m \to \infty$ for all $h \in \mathcal{H}$ and $i = 2, 3$),
(iii) $\mathbb{S}_2^{-1}(\hat{T}_1, \hat{T}_1^*) = I - T_2T_2^* - T_3T_3^* + T_2T_3T_2^*T_3^* \geq 0$, and
(iv) $\mathbb{S}_2^{-1}(\hat{T}_2, \hat{T}_2^*) = I - T_1T_1^* - T_3T_3^* + T_1T_3T_1^*T_3^* \geq 0$.

Under the additional assumption that $\|T_i\| < 1$, $i = 1, \ldots, n$, the above class of n-tuples of commuting contractions has been studied, and denoted by $\mathcal{P}^n_{p,q}(\mathcal{H})$, by Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman in [18]. It is easy to see that $\|T_i\| < 1$, $i = 1, \ldots, n$, implies that $(T_1, \ldots, T_n)$ is a pure tuple. More specifically, for every $1 \leq p < q \leq n$, it is immediate that

$$(M_{z_1}, \ldots, M_{z_n}) \in \mathcal{T}^n_{p,q}(H^2(\mathbb{D}^n)),$$

but

$$(M_{z_1}, \ldots, M_{z_n}) \notin \mathcal{P}^n_{p,q}(H^2(\mathbb{D}^n)),$$

and so

$$\mathcal{P}^n_{p,q}(H^2(\mathbb{D}^n)) \subsetneq \mathcal{T}^n_{p,q}(H^2(\mathbb{D}^n)).$$

It should be noted, however, that $\mathcal{P}^n_{p,q}(\mathcal{H})$ is a dense subset of $\mathcal{T}^n_{p,q}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. This follows from the fact that if $T \in \mathcal{T}^n(\mathcal{H})$ and $\mathbb{S}_n^{-1}(T, T^*) \geq 0$, then for any $0 < r < 1$

$$\mathbb{S}_n^{-1}(rT, rT^*) \geq 0,$$

where $rT = (rT_1, \ldots, rT_n)$.

2.4. **Transfer functions.** Our approach to isometric dilations and refined von Neumann inequality will rely on the theory of transfer functions. Let $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces, and let $U : \mathcal{E} \oplus \mathcal{H} \to \mathcal{E}_* \oplus \mathcal{H}$ be a unitary operator. Assume that

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{E} \oplus \mathcal{H} \to \mathcal{E}_* \oplus \mathcal{H}.$$

Then the *transfer function* τ_U corresponding to U is defined by

$$\tau_U(z) = A + Bz(I_{\mathcal{H}} - Dz)^{-1}C,$$

for all $z \in \mathbb{D}$. Since $\|D\| \leq 1$, and so $\|zD\| < 1$ for all $z \in \mathbb{D}$, it follows that τ_U is a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued analytic function on $\mathbb{D}$. Moreover, a standard and well-known computation (cf. [2]) yields that

$$I - \tau_U(z)^*\tau_U(z) = (1 - |z|^2)C^*(I_{\mathcal{H}} - \bar{z}D^*)^{-1}(I_{\mathcal{H}} - zD)^{-1}C, \tag{2.1}$$

for all $z \in \mathbb{D}$. In particular, $\tau_U \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ and $\|M_{\tau_U}\| \leq 1$, that is, τ_U is a contractive multiplier. We refer the reader to the monograph by Agler and McCarthy [2] for more details.

3. Dilations for finite rank tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$

Let $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$. We say that T is of *finite rank* if

$$\dim \mathcal{D}_{\hat{T}_i} < \infty,$$

for all $i = p, q$. In this section we find explicit dilation for a finite rank n-tuple of commuting contractions in $\mathcal{T}^n_{p,q}(\mathcal{H})$. Our (explicit) dilation result seems to be especially more useful in studying refined von Neumann inequality. Recall that for $T \in \mathcal{T}^n(\mathcal{H})$ and $i \in \{1, \ldots, n\}$, $\hat{T}_i$ is defined as

$$\hat{T}_i = (T_1, \ldots, T_{i-1}, T_{i+1}, T_{i+2}, \ldots, T_n) \in \mathcal{T}^{(n-1)}(\mathcal{H}).$$

Let us also introduce the following notations, which will be extensively used in the sequel. For $T \in \mathcal{T}^n(\mathcal{H})$, define

$$\hat{T}_{p,q} = (T_1, \ldots, T_{p-1}, T_{p+1}, \ldots, T_{q-1}, T_{q+1}, \ldots, T_n) \in \mathcal{T}^{(n-2)}(\mathcal{H}), \tag{3.1}$$

the $(n-2)$-tuple obtained from T by deleting T_p and T_q, and

$$\hat{T}_{pq} = (T_1, \ldots, T_{p-1}, \underbrace{T_pT_q}_{p^{th}\ place}, T_{p+1}, \ldots, T_{q-1}, T_{q+1}, \ldots, T_n) \in \mathcal{T}^{(n-1)}(\mathcal{H}), \tag{3.2}$$

the $(n-1)$-tuple obtained from T by removing T_q and replacing T_p by the product T_pT_q.

We begin with the following useful lemma on defect operators.

Lemma 3.1. *If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, then*

$$D^2_{\hat{T}_{pq}} = D^2_{\hat{T}_p} + T_q D^2_{\hat{T}_q} T_q^* = T_p D^2_{\hat{T}_p} T_p^* + D^2_{\hat{T}_q}.$$

Proof. Since by definition

$$D^2_{\hat{T}_{p,q}} = \mathbb{S}^{-1}_{n-2}(\hat{T}_{p,q}, \hat{T}^*_{p,q}),$$

it follows that

$$D^2_{\hat{T}_p} = D^2_{\hat{T}_{p,q}} - T_q D^2_{\hat{T}_{p,q}} T_q^*,$$

and

$$D^2_{\hat{T}_q} = D^2_{\hat{T}_{p,q}} - T_p D^2_{\hat{T}_{p,q}} T_p^*.$$

Then

$$\begin{aligned} D^2_{\hat{T}_{pq}} &= \mathbb{S}^{-1}_{n-1}(\hat{T}_{pq}, \hat{T}^*_{pq}) \\ &= D^2_{\hat{T}_{p,q}} - T_pT_q D^2_{\hat{T}_{p,q}} T_p^* T_q^* \\ &= D^2_{\hat{T}_{p,q}} - T_p D^2_{\hat{T}_{p,q}} T_p^* + T_p(D^2_{\hat{T}_{p,q}} - T_q D^2_{\hat{T}_{p,q}} T_q^*)T_p^*, \end{aligned}$$

that is

$$D^2_{\hat{T}_{pq}} = D^2_{\hat{T}_q} + T_p D^2_{\hat{T}_p} T_p^*, \tag{3.3}$$

and similarly
$$D^2_{\hat{T}_{pq}} = D^2_{\hat{T}_p} + T_q D^2_{\hat{T}_q} T_q^*.$$
This completes the proof of the lemma. □

Therefore, if $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, then it follows clearly from the above lemma that the map
$$U : \{D_{\hat{T}_p} h \oplus D_{\hat{T}_q} T_q^* h : h \in \mathcal{H}\} \to \{D_{\hat{T}_p} T_p^* h \oplus D_{\hat{T}_q} h : h \in \mathcal{H}\}$$
defined by
$$U(D_{\hat{T}_p} h, D_{\hat{T}_q} T_q^* h) = (D_{\hat{T}_p} T_p^* h, D_{\hat{T}_q} h),$$
for all $h \in \mathcal{H}$, is an isometry. In addition, if
$$\dim \mathcal{D}_{\hat{T}_i} < \infty,$$
for all $i = p, q$, then U extends to a unitary on $\mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q}$, which we denote again by U. This implies the first part of the lemma below.

Lemma 3.2. *If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$ is a finite rank tuple, then there exists a unitary $U \in \mathcal{B}(\mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q})$ such that*
$$U(D_{\hat{T}_p} h, D_{\hat{T}_q} T_q^* h) = (D_{\hat{T}_p} T_p^* h, D_{\hat{T}_q} h),$$
for all $h \in \mathcal{H}$. Moreover, if
$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q} \to \mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q},$$
then
$$D_{\hat{T}_p} T_p^* = A D_{\hat{T}_p} + \sum_{i=0}^{\infty} B D^i C D_{\hat{T}_p} T_q^{*i+1},$$
where the series converges in the strong operator topology.

Proof. We only need to prove the second part. Let $h \in \mathcal{H}$. Using
$$U(D_{\hat{T}_p} h, D_{\hat{T}_q} T_q^* h) = (D_{\hat{T}_p} T_p^* h, D_{\hat{T}_q} h),$$
we obtain
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D_{\hat{T}_p} h \\ D_{\hat{T}_q} T_q^* h \end{bmatrix} = \begin{bmatrix} D_{\hat{T}_p} T_p^* h \\ D_{\hat{T}_q} h \end{bmatrix}.$$
Then
$$D_{\hat{T}_p} T_p^* h = A D_{\hat{T}_p} h + B D_{\hat{T}_q} T_q^* h,$$
and
$$D_{\hat{T}_q} h = C D_{\hat{T}_p} h + D D_{\hat{T}_q} T_q^* h.$$
Repeatedly resolving the former equation for $D_{\hat{T}_p} T_p^* h$ in the latter equation, we obtain
$$D_{\hat{T}_p} T_p^* h = A D_{\hat{T}_p} h + \sum_{i=1}^{m} B D^i C D_{\hat{T}_p} T_q^{*(i+1)} h + B D^{m+1} D_{\hat{T}_q} T_q^{*(m+2)} h,$$
for all $h \in \mathcal{H}$ and $m \geq 1$. The proof now follows from the fact that $T_q^{*m} h \to 0$ as $m \to \infty$ and $\|D\| \leq 1$. □

The proof of the second half of Lemma 3.2, motivated by [13], will play an important role in what follows.

Theorem 3.3. *If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$ is a finite rank tuple, then there exist an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$ and an inner function $\varphi \in H^\infty_{\mathcal{B}(\mathcal{D}_{\hat{T}_p})}(\mathbb{D})$ such that*

$$\Pi T_i^* = \begin{cases} M_{z_i}^* \Pi & \text{if } 1 \leq i < p, \\ M_{\Phi_p}^* \Pi & \text{if } i = p, \\ M_{z_{i-1}}^* \Pi & \text{if } p < i \leq n, \end{cases}$$

where

$$\Phi_p(\boldsymbol{z}) = \varphi(z_{q-1}),$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$. In particular, T on $\mathcal{H}$ dilates to the n-tuple of commuting isometries

$$(M_{z_1}, \dots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_p}, \dots, M_{z_{n-1}})$$

on $H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$.

Proof. Since $\hat{T}_p \in \mathbb{S}_{n-1}(\mathcal{H})$ is a pure contraction, we have

$$\Pi T_i^* = \begin{cases} M_{z_i}^* \Pi & \text{if } 1 \leq i < p, \\ M_{z_{i-1}}^* \Pi & \text{if } p < i \leq n, \end{cases}$$

where $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$, defined by

$$(\Pi h)(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^{n-1}} \boldsymbol{z}^{\boldsymbol{k}} D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{k}} h,$$

for $\boldsymbol{z} \in \mathbb{D}^{n-1}$ and $h \in \mathcal{H}$, is the canonical isometry corresponding to $\hat{T}_p$ (see Theorem 2.1). We prove that

$$\Pi T_p^* = M_{\Phi_p} \Pi,$$

for some one-variable (in z_{q-1}) inner function $\Phi_p \in H^\infty_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$. To this end, consider $h \in \mathcal{H}$, $\eta \in \mathcal{D}_{\hat{T}_p}$ and $\boldsymbol{k} \in \mathbb{Z}_+^{n-1}$. Then

$$\begin{aligned} \langle \Pi T_p^* h, \boldsymbol{z}^{\boldsymbol{k}} \eta \rangle &= \langle \sum_{\boldsymbol{l} \in \mathbb{Z}_+^{n-1}} \boldsymbol{z}^{\boldsymbol{l}} D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{l}} T_p^* h, \boldsymbol{z}^{\boldsymbol{k}} \eta \rangle \\ &= \langle D_{\hat{T}_p} T_p^* \hat{T}_p^{*\boldsymbol{k}} h, \eta \rangle. \end{aligned}$$

Next, consider the unitary

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q} \to \mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q}$$

as in Lemma 3.2. Let

$$\Phi_p(\boldsymbol{z}) = \tau_{U^*}(z_{q-1}),$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$, where

$$\tau_{U^*}(z) = A^* + zC^*(I_{\mathcal{D}_{\hat{T}_q}} - zD^*)^{-1} B^*,$$

for all $z \in \mathbb{D}$, is the transfer function corresponding to the unitary map U^*. Since dim $\mathcal{D}_{\hat{T}_p} < \infty$, the equality (2.1) implies that τ_U is an inner multiplier on $\mathbb{D}$. Also we compute

$$\begin{aligned}
\langle M_{\Phi_p}^* \Pi h, \boldsymbol{z}^{\boldsymbol{k}} \eta \rangle &= \langle \Pi h, \Phi_p(\boldsymbol{z})(\boldsymbol{z}^{\boldsymbol{k}}\eta)\rangle \\
&= \langle \sum_{\boldsymbol{l} \in \mathbb{Z}_+^{n-1}} \boldsymbol{z}^{\boldsymbol{l}} D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{l}} h, (A^* + C^* \sum_{m=0}^{\infty} D^{*m} B^* z_{q-1}^{m+1}) \boldsymbol{z}^{\boldsymbol{k}} \eta \rangle \\
&= \langle D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{k}} h, A^* \eta \rangle + \sum_{m=0}^{\infty} \langle D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{k}} T_q^{*m+1} h, C^* D^{*m} B^* \eta \rangle \\
&= \langle A D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{k}} h, \eta \rangle + \sum_{m=0}^{\infty} \langle B D^m C D_{\hat{T}_p} \hat{T}_p^{*\boldsymbol{k}} T_q^{*m+1} h, \eta \rangle \\
&= \langle (A D_{\hat{T}_p} + \sum_{m=0}^{\infty} B D^m C D_{\hat{T}_p} T_q^{*m+1}) \hat{T}_p^{*\boldsymbol{k}} h, \eta \rangle,
\end{aligned}$$

and so, by Lemma 3.2

$$\langle M_{\Phi_p}^* \Pi h, \boldsymbol{z}^{\boldsymbol{k}} \eta \rangle = \langle D_{\hat{T}_p} T_p^* \hat{T}_p^{*\boldsymbol{k}} h, \eta \rangle.$$

Thus

$$\Pi T_p^* = M_{\Phi_p}^* \Pi.$$

This completes the proof. ◻

The discussion in Subsection 2.1 gives another way to describe the above dilation theorem: If $T \in \mathcal{T}_{p,q}^n(\mathcal{H})$, then T and

$$(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_{p-1}}|_{\mathcal{Q}}, P_{\mathcal{Q}} M_{\Phi_p}|_{\mathcal{Q}}, P_{\mathcal{Q}} M_{z_p}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_{n-1}}|_{\mathcal{Q}})$$

on $\mathcal{Q}$ are jointly unitarily equivalent, where

$$\mathcal{Q} = \text{ran } \Pi \subseteq H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1}),$$

is a joint invariant subspace for

$$(M_{z_1}^*, \ldots, M_{z_{p-1}}^*, M_{\Phi_p}^*, M_{z_p}^*, \ldots, M_{z_{n-1}}^*).$$

A natural question arises about the isometric dilation: What can be said if the assumption of finite dimensionality in Theorem 3.3 is removed? In the general case, the above ideas allow one to prove that Φ_p is a contractive multiplier. We proceed as follows: Let $\mathcal{D}_{\hat{T}_p}$ or $\mathcal{D}_{\hat{T}_q}$ is an infinite dimensional Hilbert space. Let $\mathcal{D}$ be an infinite dimensional Hilbert space such that the isometry

$$U : \{D_{\hat{T}_p} h \oplus D_{\hat{T}_q} T_q^* h : h \in \mathcal{H}\} \oplus \{0_{\mathcal{D}}\} \to \{D_{\hat{T}_p} T_p^* h \oplus D_{\hat{T}_q} h : h \in \mathcal{H}\} \oplus \{0_{\mathcal{D}}\}$$

defined by

$$U(D_{\hat{T}_p} h, D_{\hat{T}_q} T_q^* h, 0_{\mathcal{D}}) = (D_{\hat{T}_p} T_p^* h, D_{\hat{T}_q} h, 0_{\mathcal{D}}),$$

for $h \in \mathcal{H}$, extends to a unitary, again denoted by U, on

$$\mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q} \oplus \mathcal{D}.$$

Then the same conclusion as in Lemma 3.2 holds for the unitary

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{D}_{\hat{T}_p} \oplus (\mathcal{D}_{\hat{T}_q} \oplus \mathcal{D})).$$

However, in this case, the transfer function τ_{U^*} is a contractive analytic function. Then following the same line of argument as in the proof of Theorem 3.3, we have:

Theorem 3.4. *If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, then there exist an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$ and a contractive multiplier $\varphi \in H^\infty_{\mathcal{B}(\mathcal{D}_{\hat{T}_p})}(\mathbb{D})$ such that*

$$\Pi T_i^* = \begin{cases} M^*_{z_i}\Pi & \text{if } 1 \le i < p, \\ M^*_{\Phi_p}\Pi & \text{if } i = p, \\ M^*_{z_{i-1}}\Pi & \text{if } p < i \le n, \end{cases}$$

where $\Phi_p(\boldsymbol{z}) = \varphi(z_{q-1})$ for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$.

In Theorem 5.3 we present a sharper version of the above theorem: Every n-tuple in $\mathcal{T}^n_{p,q}(\mathcal{H})$ has an isometric dilation. This will require a completely different method. As is discussed in the following sections, the finite rank assumption in Theorem 3.3 will turn out to be useful for refined von Neumann inequality.

4. von Neumann inequality for finite rank tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$

In this section, we use the isometric dilations developed above to prove the von Neumann inequality for finite rank tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

First we recall the notion of completely non-unitary contractions [24]. A contraction T on $\mathcal{H}$ is said to be *completely non-unitary* if there is no non-zero closed reducing subspace $\mathcal{S} \subseteq \mathcal{H}$ for T such that $T|_{\mathcal{S}}$ is a unitary operator. This notion has proved to be useful in the following sense: If $T \in \mathcal{T}^1(\mathcal{H})$, then there exists a unique decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ of $\mathcal{H}$ reducing T, such that $T|_{\mathcal{H}_u}$ is unitary and $T|_{\mathcal{H}_c}$ is completely non-unitary. We therefore have the *canonical decomposition* of T as:

$$T = \begin{bmatrix} T|_{\mathcal{H}_u} & 0 \\ 0 & T|_{\mathcal{H}_c} \end{bmatrix}.$$

We need first the following result noted in [13, Proposition 4.2].

Proposition 4.1. *Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a unitary matrix on $\mathcal{H} \oplus \mathcal{K}$ and let $A = \begin{bmatrix} A_u & 0 \\ 0 & A_c \end{bmatrix} \in \mathcal{B}(\mathcal{H}_u \oplus \mathcal{H}_c)$ be the canonical decomposition of A into the unitary part A_u on $\mathcal{H}_u$ and the completely non-unitary part A_c on $\mathcal{H}_c$. Then $U' = \begin{bmatrix} A_c & B \\ C|_{\mathcal{H}_c} & D \end{bmatrix}$ is a unitary operator on $\mathcal{H}_c \oplus \mathcal{K}$ and*

$$\tau_U(z) = \begin{bmatrix} A_u & 0 \\ 0 & \tau_{U'}(z) \end{bmatrix} \in \mathcal{B}(\mathcal{H}_u \oplus \mathcal{H}_c) \qquad (z \in \mathbb{D}).$$

Now we turn to the distinguished varieties in $\mathbb{D}^2$ [3]. Recall that a non-empty set V in $\mathbb{C}^2$ is a *distinguished variety* if there is a polynomial $p \in \mathbb{C}[z_1, z_2]$ such that

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : p(z_1, z_2) = 0\},$$

and V exits the bidisc through the distinguished boundary, that is,

$$\overline{V} \cap \partial\mathbb{D}^2 = \overline{V} \cap (\partial\mathbb{D} \times \partial\mathbb{D}).$$

Here $\partial\mathbb{D}^2$ and $\partial\mathbb{D} \times \partial\mathbb{D}$ denote the boundary and the distinguished boundary of the bidisc respectively, and $\overline{V}$ is the closure of V in $\overline{\mathbb{D}^2}$. We denote by ∂V the set $\overline{V} \cap \partial\mathbb{D}^2$, the boundary of V within the zero set of the polynomial p and $\overline{\mathbb{D}^2}$.

In the seminal paper [3], Agler and McCarthy characterized distinguished varieties as follows: Let $V \subseteq \mathbb{C}^2$. Then V is a distinguished variety if and only if there exists a rational matrix inner function $\Psi \in H^\infty_{\mathcal{B}(\mathbb{C}^m)}(\mathbb{D})$, for some $m \geq 1$, such that

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\Psi(z_1) - z_2 I_{\mathbb{C}^m}) = 0\}.$$

In the following result, we restrict ourselves to the case $\mathcal{T}^n_{1,2}(\mathcal{H})$, for simplicity. The general case can be dealt with using a similar argument.

Theorem 4.2. *If $T \in \mathcal{T}^n_{1,2}(\mathcal{H})$ is a finite rank operator, then there exists an algebraic variety V in $\overline{\mathbb{D}}^n$ such that for all $p \in \mathbb{C}[z_1, \ldots, z_n]$, the following holds:*

$$\|p(T)\| \leq \sup_{\boldsymbol{z} \in V} |p(\boldsymbol{z})|.$$

If, in addition, T_1 is a pure contraction, then there exists a distinguished variety V' in $\mathbb{D}^2$ such that

$$V = V' \times \mathbb{D}^{n-2} \subseteq \mathbb{D}^n.$$

Proof. Let $(M_{\Phi_1}, M_{z_1}, \ldots, M_{z_{n-1}})$ on $H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1})$ be the isometric dilation of T provided by Theorem 3.3, where $\Phi_1 \in H^\infty_{\mathcal{B}(\mathcal{D}_{\hat{T}_1})}(\mathbb{D})$ is the inner multiplier given by

$$\Phi_1 = \tau_{U^*},$$

and

$$U^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \in \mathcal{B}(\mathcal{D}_{\hat{T}_1} \oplus \mathcal{D}_{\hat{T}_2}),$$

is the unitary as in Lemma 3.2. Let

$$A^* = \begin{bmatrix} A_u^* & 0 \\ 0 & A_c^* \end{bmatrix} \in \mathcal{B}(\mathcal{D}_u \oplus \mathcal{D}_c),$$

be the canonical decomposition of A^* on

$$\mathcal{D}_{\hat{T}_1} = \mathcal{D}_u \oplus \mathcal{D}_c,$$

into the unitary part A_u^* on $\mathcal{D}_u$ and the completely non-unitary part A_c^* on $\mathcal{D}_c$. If we set

$$U_c^* = \begin{bmatrix} A_c^* & C^* \\ B^*|_{\mathcal{D}_c} & D^* \end{bmatrix} \in \mathcal{B}(\mathcal{D}_c \oplus \mathcal{D}_{\hat{T}_2}),$$

then Proposition 4.1 implies that

$$\Phi_1(z) = \begin{bmatrix} \Phi_u(z) & 0 \\ 0 & \Phi_c(z) \end{bmatrix},$$

where

$$\Phi_u(z) \equiv A_u^*,$$

and

$$\Phi_c(z) = \tau_{U_c^*},$$

for all $z \in \mathbb{D}$. Let

$$V_u = \{(z,w) \in \bar{\mathbb{D}}^2 : \det(zI_{\mathcal{D}_u} - \Phi_u(w)) = 0\} \times \mathbb{D}^{n-2}$$

and

$$V_c = \{(z,w) \in \mathbb{D}^2 : \det(zI_{\mathcal{D}_c} - \Phi_c(w)) = 0\} \times \mathbb{D}^{n-2}.$$

Since $\Phi_c \in H^\infty_{\mathcal{B}(\mathcal{D}_c)}(\mathbb{D})$ is a rational matrix inner function (cf. page 138, [3]), the characterization result of distinguished varieties by Agler and McCarthy (Theorem 1.12 in [3]) implies that

$$\{(z,w) \in \mathbb{D}^2 : \det(zI_{\mathcal{D}_c} - \Phi_c(w)) = 0\},$$

is a distinguished variety in $\mathbb{D}^2$. Now let $p \in \mathbb{C}[z_1, \ldots, z_n]$. Since the discussion following Theorem 3.3 implies that T on $\mathcal{H}$ and

$$(P_{\mathcal{Q}} M_{\Phi_1}|_{\mathcal{Q}}, P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_{n-1}}|_{\mathcal{Q}}),$$

on

$$\mathcal{Q} = \operatorname{ran} \Pi \subseteq H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1}), \tag{4.1}$$

are unitarily equivalent, it follows that

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} = \|P_{\mathcal{Q}} p(M_{\Phi_1}, M_{z_1}, \ldots, M_{z_{n-1}})|_{\mathcal{Q}}\|_{\mathcal{B}(\mathcal{Q})},$$

and so

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \|p(M_{\Phi_1}, M_{z_1}, \ldots, M_{z_{n-1}})\|_{\mathcal{B}(H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1}))}.$$

But

$$\|p(M_{\Phi_1}, M_{z_1}, \ldots, M_{z_{n-1}})\|_{\mathcal{B}(H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1}))} = \|M_{p(\Phi_1(z_1), z_1 I_{\mathcal{D}_{\hat{T}_1}}, \ldots, z_{n-1} I_{\mathcal{D}_{\hat{T}_1}})}\|_{\mathcal{B}(H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1}))},$$

and

$$\|M_{p(\Phi_1(z_1), z_1 I_{\mathcal{D}_{\hat{T}_1}}, \ldots, z_{n-1} I_{\mathcal{D}_{\hat{T}_1}})}\|_{\mathcal{B}(H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1}))} \leq \|p(\Phi_1(z_1), z_1 I_{\mathcal{D}_{\hat{T}_1}}, \ldots, z_{n-1} I_{\mathcal{D}_{\hat{T}_1}})\|_{H^\infty_{\mathcal{B}(\mathcal{D}_{\hat{T}_1})}(\mathbb{D}^{n-1})}.$$

Clearly, the right side is equal to

$$\sup_{\theta_1, \ldots, \theta_{n-1}} \|p(\Phi_1(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_{\hat{T}_1}}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_{\hat{T}_1}})\|_{\mathcal{B}(\mathcal{D}_{\hat{T}_1})}.$$

Hence we have

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\theta_1, \ldots, \theta_{n-1}} \|p(\Phi_1(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_{\hat{T}_1}}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_{\hat{T}_1}})\|_{\mathcal{B}(\mathcal{D}_{\hat{T}_1})}.$$

Now for each $\theta_1, \ldots, \theta_{n-1}$, the orthogonal decomposition of

$$\Phi_1(e^{i\theta_1}) = \Phi_u(e^{i\theta_1}) \oplus \Phi_c(e^{i\theta_1}),$$

on $\mathcal{D}_{\hat{T}_1} = \mathcal{D}_u \oplus \mathcal{D}_c$ applied to

$$p(\Phi_1(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_{\hat{T}_1}}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_{\hat{T}_1}}) \in \mathcal{B}(\mathcal{D}_{\hat{T}_1}),$$

shows that

$$\begin{aligned}\|p(\Phi_1(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_{\hat{T}_1}}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_{\hat{T}_1}})\|_{\mathcal{B}(\mathcal{D}_{\hat{T}_1})} = \max\{&\|p(\Phi_u(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_u}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_u})\|_{\mathcal{B}(\mathcal{D}_u)}, \\ &\|p(\Phi_c(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_c}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_c})\|_{\mathcal{B}(\mathcal{D}_c)}\}.\end{aligned}$$

Note further that

$$\|p(\Phi_s(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_s}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_s})\|_{\mathcal{B}(\mathcal{D}_s)} = \sup_{\lambda \in \sigma(\Phi_s(e^{i\theta_1}))} |p(\lambda, e^{i\theta_1}, \ldots, e^{i\theta_{n-1}})|,$$

for all $s = u, c$, and hence

$$\|p(\Phi_1(e^{i\theta_1}), e^{i\theta_1} I_{\mathcal{D}_{\hat{T}_1}}, \ldots, e^{i\theta_{n-1}} I_{\mathcal{D}_{\hat{T}_1}})\|_{\mathcal{B}(\mathcal{D}_{\hat{T}_1})} \leq \sup_{\lambda \in \sigma(\Phi_u(e^{i\theta_1})) \cup \sigma(\Phi_c(e^{i\theta_1}))} |p(\lambda, e^{i\theta_1}, \ldots, e^{i\theta_{n-1}})|.$$

Consequently we have

$$\begin{aligned}\|p(T)\|_{\mathcal{B}(\mathcal{H})} &\leq \sup_{\theta_1, \ldots, \theta_{n-1}} \{|p(\lambda, e^{i\theta_1}, \ldots, e^{i\theta_{n-1}})| : \lambda \in \sigma(\Phi_u(e^{i\theta_1})) \cup \sigma(\Phi_c(e^{i\theta_1}))\} \\ &= \sup_{\boldsymbol{z} \in \partial V_u \cup \partial V_c} |p(\boldsymbol{z})|,\end{aligned}$$

and hence, by continuity

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_V,$$

where

$$V = V_u \cup V_c.$$

This proves the first part of the theorem. Assume now that T_1 is a pure contraction. It is enough to prove that V_u is an empty set. Since $\Phi_u(z) \equiv A_u^*$, $z \in \mathbb{D}$, and A_u is a unitary on $\mathcal{D}_u$, this is equivalent to proving that

$$\mathcal{D}_u = \{0\},$$

which is further equivalent to the condition that A^* is completely non-unitary. First, we observe that (see (4.1))

$$\mathcal{Q} \subseteq H^2_{\mathcal{D}_c}(\mathbb{D}^{n-1}).$$

Indeed, let $g \in H^2_{\mathcal{D}_u}(\mathbb{D}^{n-1})$, $m \in \mathbb{Z}_+$ and set $g_m = M_{\Phi_u}^{*m} g$. Then

$$g_m = A_u^{*m} g \in H^2_{\mathcal{D}_u}(\mathbb{D}^{n-1}),$$

and

$$M_{\Phi_u}^m g_m = g.$$

Now, if $f \in \mathcal{Q}$, then clearly

$$\begin{aligned}\langle g, f \rangle &= \langle M_{\Phi_u}^m g_m, f \rangle \\ &= \langle g_m, T_1^{*m} f \rangle,\end{aligned}$$

and so

$$\begin{aligned}|\langle g, f\rangle| &\leq \|g_m\|\|T_1^{*m} f\| \\ &= \|g\|\|T_1^{*m} f\|.\end{aligned}$$

Since T_1 is a pure contraction, it follows that

$$\langle g, f\rangle = 0,$$

and therefore $\mathcal{Q} \subseteq H^2_{\mathcal{D}_c}(\mathbb{D}^{n-1})$. Finally, on the one hand, by the property of the isometric dilation we have

$$\bigvee_{\boldsymbol{k}\in\mathbb{Z}_+^{n-1}} M_{\boldsymbol{z}}^{\boldsymbol{k}} \mathcal{Q} = H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1}),$$

and on the other hand $H^2_{\mathcal{D}_c}(\mathbb{D}^{n-1}) \subseteq H^2_{\mathcal{D}_{\hat{T}_1}}(\mathbb{D}^{n-1})$ is a joint reducing subspace for $(M_{z_1}, \ldots, M_{z_{n-1}})$. Hence $H^2_{\mathcal{D}_u}(\mathbb{D}^{n-1}) = \{0\}$ and therefore $\mathcal{D}_u = \{0\}$. The theorem is proved. ☐

As we have pointed out before, the above von Neumann inequality for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$ is finer and conceptually different (under the finite rank assumption) from the one obtained by Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman [18].

5. Dilations for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$

In this section we again investigate dilations for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$. We prove the validity of Statements 1* and 2 for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$ without any additional assumption on the defect spaces $\mathcal{D}_{\hat{T}_p}$ and $\mathcal{D}_{\hat{T}_q}$.

Recall that (see Equations (3.1) and (3.2)) for $T \in \mathcal{T}^n(\mathcal{H})$, we denote

$$\hat{T}_{p,q} = (T_1, \ldots, T_{p-1}, T_{p+1}, \ldots, T_{q-1}, T_{q+1}, \ldots, T_n) \in \mathcal{T}^{(n-2)}(\mathcal{H}),$$

the $(n-2)$-tuple obtained from T by deleting T_p and T_q, and

$$\hat{T}_{pq} = (T_1, \ldots, T_{p-1}, \underbrace{T_p T_q}_{p^{th}\ place}, T_{p+1}, \ldots, T_{q-1}, T_{q+1}, \ldots, T_n) \in \mathcal{T}^{(n-1)}(\mathcal{H}),$$

the $(n-1)$-tuple obtained from T by removing T_q and replacing T_p by the product $T_p T_q$. We begin with a simple but important observation.

Lemma 5.1. *If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, then $\hat{T}_{pq}$ is a pure tuple and $\hat{T}_{pq} \in \mathbb{S}_{n-1}(\mathcal{H})$.*

Proof. Since $T_p T_q = T_q T_p$ and T_q is a pure contraction, it follows that $T_p T_q$ is a pure contraction, and hence $\hat{T}_{pq}$ is a pure tuple. On the other hand, by (3.3) we have

$$D^2_{\hat{T}_{pq}} = D^2_{\hat{T}_q} + T_p D^2_{\hat{T}_p} T_p^*,$$

and therefore,

$$D^2_{\hat{T}_{pq}} \geq 0,$$

as $\hat{T}_p, \hat{T}_q \in \mathbb{S}_{n-1}(\mathcal{H})$. This completes the proof of the lemma. ☐

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert spaces, and let

$$U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$

be a unitary operator on $\mathcal{E}_1 \oplus \mathcal{E}_2$. Then the $\mathcal{B}(\mathcal{E}_1)$-valued transfer function τ_U on $\mathbb{D}$, defined by (see Subsection 2.4)

$$\tau_U(z) = A + zBC,$$

satisfies the equality (see (2.1))

$$I - \tau_U(z)^* \tau_U(z) = (1 - |z|^2) C^* C,$$

for all $z \in \mathbb{D}$. In particular, $\tau_U \in H^\infty_{\mathcal{B}(\mathcal{E}_1)}(\mathbb{D})$ is an inner function. Now if $1 \leq p \leq n$ and

$$\Phi(\boldsymbol{z}) = \tau_U(z_p),$$

for all $\boldsymbol{z} \in \mathbb{D}^n$, then $\Phi \in H^\infty_{\mathcal{B}(\mathcal{E}_1)}(\mathbb{D}^n)$ is an inner polynomial in z_p of degree at most 1. This point of view will be used in what follows to develop the dilation theory for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

We now proceed to give an explicit description of isometric dilations of tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$. Let $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$. Then, by the previous lemma, $\hat{T}_{pq} \in \mathbb{S}_{n-1}(\mathcal{H}) \cap \mathcal{T}^{(n-1)}(\mathcal{H})$ is a pure tuple. Let

$$\mathcal{D}_{\hat{T}_{pq}} = \overline{ran}\, \mathbb{S}_{n-1}(\hat{T}_{pq}, \hat{T}^*_{pq}),$$

and let $\Pi_{pq} : \mathcal{H} \to H^2_{\mathcal{D}_{\hat{T}_{pq}}}(\mathbb{D}^{n-1})$ be the canonical isometry corresponding to $\hat{T}_{pq}$ (see Theorem 2.1). Then

$$\Pi_{pq} R_i^* = (M_{z_i} \otimes I_{\mathcal{D}_{\hat{T}_{pq}}})^* \Pi_{pq}, \tag{5.1}$$

for all $i = 1, \ldots, n-1$, where

$$R_i = \begin{cases} T_i & \text{if } 1 \leq i < q, i \neq p, \\ T_p T_q & \text{if } i = p, \\ T_{i+1} & \text{if } q \leq i \leq n-1. \end{cases}$$

In other words, $(R_1, \ldots, R_{n-1}) = \hat{T}_{pq}$, that is

$$(R_1, \ldots, R_{n-1}) = (T_1, \ldots, T_{p-1}, \underbrace{T_p T_q}_{p^{th}\ place}, T_{p+1}, \ldots, T_{q-1}, T_{q+1}, \ldots, T_n),$$

on $\mathcal{H}$ dilates to

$$(M_{z_1} \otimes I_{\mathcal{D}_{\hat{T}_{pq}}}, \ldots, M_{z_{n-1}} \otimes I_{\mathcal{D}_{\hat{T}_{pq}}}),$$

on $H^2_{\mathcal{D}_{\hat{T}_{pq}}}(\mathbb{D}^{n-1})$ via the canonical isometry $\Pi_{pq} : \mathcal{H} \to H^2_{\mathcal{D}_{\hat{T}_{pq}}}(\mathbb{D}^{n-1})$. Now let $\mathcal{E}$ be a Hilbert space, and let $V : \mathcal{D}_{\hat{T}_{pq}} \to \mathcal{E}$ be an isometry. Let

$$\Pi_{V,pq} = (I_{H^2(\mathbb{D}^{n-1})} \otimes V) \circ \Pi_{pq} \in \mathcal{B}(\mathcal{H}, H^2_{\mathcal{E}}(\mathbb{D}^{n-1})). \tag{5.2}$$

Then $\Pi_{V,pq} : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D}^{n-1})$ is an isometry and

$$\Pi_{V,pq} R_i^* = (M_{z_i} \otimes I_{\mathcal{E}})^* \Pi_{V,pq}, \tag{5.3}$$

for all $i = 1, \ldots, n-1$. So $\hat{T}_{pq}$ on $\mathcal{H}$ dilates to $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_{n-1}} \otimes I_{\mathcal{E}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n-1})$ via the isometry $\Pi_{V,pq}$. Now we are ready to prove the key lemma.

Lemma 5.2. *Let $\mathcal{H}$ and $\mathcal{E}$ be Hilbert spaces, let $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, and let V and $\Pi_{V,pq}$ be as above. Let F_1 and F_2 be bounded operators on $\mathcal{H}$, and let $\mathcal{F}_i = \overline{ran}\, F_i$, $i = 1, 2$. Let*

$$U_i = \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{F}_i \to \mathcal{E} \oplus \mathcal{F}_i,$$

be a unitary operator, $i = 1, 2$. If

$$U_1(VD_{\hat{T}_{pq}}h, F_1 T_p^* T_q^* h) = (VD_{\hat{T}_{pq}} T_p^* h, F_1 h),$$

and

$$U_2(VD_{\hat{T}_{pq}}h, F_2 T_p^* T_q^* h) = (VD_{\hat{T}_{pq}} T_q^* h, F_2 h),$$

for all $h \in \mathcal{H}$, then

$$\Pi_{V,pq} T_p^* = M^*_{\Phi_1} \Pi_{V,pq},$$

and

$$\Pi_{V,pq} T_q^* = M^*_{\Phi_2} \Pi_{V,pq},$$

where

$$\Phi_i(\boldsymbol{z}) = A_i^* + z_p C_i^* B_i^* \qquad (\boldsymbol{z} \in \mathbb{D}^{n-1}),$$

is the $\mathcal{B}(\mathcal{E})$-valued one variable transfer function of U_i^*, $i = 1, 2$. In particular, $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^{n-1})$, $i = 1, 2$, is an inner polynomial in z_p of degree at most 1.

Proof. Because of the symmetric roles of T_p and T_q, we only prove that $\Pi_{V,pq} T_p^* = M^*_{\Phi_1} \Pi_{V,pq}$. Let $h \in \mathcal{H}$, $\boldsymbol{k} \in \mathbb{Z}^{n-1}_+$ and let $\eta \in \mathcal{E}$. Using the definition of Π_{pq}, we have

$$\begin{aligned}
\langle \Pi_{V,pq} T_p^* h, \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta \rangle &= \langle (I_{H^2(\mathbb{D}^{n-1})} \otimes V) \Pi_{pq} T_p^* h, \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta \rangle \\
&= \langle (I_{H^2(\mathbb{D}^{n-1})} \otimes V) \sum_{\boldsymbol{l} \in \mathbb{Z}^{n-1}_+} \boldsymbol{z}^{\boldsymbol{l}} \otimes D_{\hat{T}_{pq}} \hat{T}^{*\boldsymbol{l}}_{pq} T_p^* h, \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta \rangle \\
&= \langle V D_{\hat{T}_{pq}} \hat{T}^{*\boldsymbol{k}}_{pq} T_p^* h, \eta \rangle.
\end{aligned}$$

Also since

$$U_1(VD_{\hat{T}_{pq}}h, F_1 T_p^* T_q^* h) = (VD_{\hat{T}_{pq}} T_p^* h, F_1 h),$$

for $h \in \mathcal{H}$, we find that

$$VD_{\hat{T}_{pq}} T_p^* = A_1 V D_{\hat{T}_{pq}} + B_1 F_1 T_p^* T_q^*,$$

and

$$F_1 = C_1 V D_{\hat{T}_{pq}}.$$

Putting this together yields

$$VD_{\hat{T}_{pq}} T_p^* = A_1 V D_{\hat{T}_{pq}} + B_1 C_1 V D_{\hat{T}_{pq}} T_p^* T_q^*,$$

and so

$$\begin{aligned}\langle M_{\Phi_1}^* \Pi_{V,pq} h, \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta\rangle &= \langle \Pi_{V,pq} h, M_{\Phi_1}(\boldsymbol{z}^{\boldsymbol{k}} \otimes \eta)\rangle \\ &= \langle (I_{H^2(\mathbb{D}^{n-1})} \otimes V) \sum_{\boldsymbol{l} \in \mathbb{Z}_+^{n-1}} \boldsymbol{z}^{\boldsymbol{l}} \otimes D_{\hat{T}_{pq}} \hat{T}_{pq}^{*\boldsymbol{l}} h, (A_1^* + z_p C_1^* B_1^*)(\boldsymbol{z}^{\boldsymbol{k}} \otimes \eta)\rangle \\ &= \langle A_1 V D_{\hat{T}_{pq}} \hat{T}^{*\boldsymbol{k}} h, \eta\rangle + \langle B_1 C_1 V D_{\hat{T}_{pq}} \hat{T}_{pq}^{*\boldsymbol{k}} T_p^* T_q^* h, \eta\rangle \\ &= \langle V D_{\hat{T}_{pq}} \hat{T}_{pq}^{*\boldsymbol{k}} T_p^* h, \eta\rangle,\end{aligned}$$

and thus $\Pi_{V,pq} T_p^* = M_{\Phi_1}^* \Pi_{V,pq}$ as required. The final claim follows easily from the paragraph following Lemma 5.1. This completes the proof of the lemma. □

Now we are ready to prove the main dilation result of this section.

Theorem 5.3. *Let $\mathcal{H}$ be a Hilbert space, and let $T = (T_1, \ldots, T_n) \in \mathcal{T}_{p,q}^n(\mathcal{H})$. Then there exist a Hilbert space $\mathcal{E}$ and an isometry $\Pi : \mathcal{H} \to H_{\mathcal{E}}^2(\mathbb{D}^{n-1})$ such that*

$$\Pi T_i^* = \begin{cases} M_{z_i}^* \Pi & \text{if } 1 \leq i < q, i \neq p, \\ M_{\Phi_i}^* \Pi & \text{if } i = p, q, \\ M_{z_{i-1}}^* \Pi & \text{if } q < i \leq n, \end{cases}$$

where Φ_p and Φ_q in $H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D}^{n-1})$ are inner polynomials in z_p of degree at most one and

$$\Phi_p(\boldsymbol{z})\Phi_q(\boldsymbol{z}) = \Phi_q(\boldsymbol{z})\Phi_p(\boldsymbol{z}) = z_p I_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$. In particular, $(T_1, \ldots, T_n) \in \mathcal{T}_{p,q}^n(\mathcal{H})$ dilates to the isometric tuple

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_{p+1}}, \ldots, M_{z_{q-1}}, M_{\Phi_q}, M_{z_q}, \ldots, M_{z_{n-1}}),$$

on $H_{\mathcal{E}}^2(\mathbb{D}^{n-1})$ via the isometry $\Pi : \mathcal{H} \to H_{\mathcal{E}}^2(\mathbb{D}^{n-1})$.

Proof. Using the identity in (3.3), we have

$$\begin{aligned} D_{\hat{T}_{pq}}^2 &= D_{\hat{T}_q}^2 + T_p D_{\hat{T}_p}^2 T_p^* \\ &= D_{\hat{T}_p}^2 + T_q D_{\hat{T}_q}^2 T_q^*, \end{aligned}$$

and then, for each $h \in \mathcal{H}$, we have

$$\begin{aligned} \|D_{\hat{T}_{pq}} h\|^2 &= \|D_{\hat{T}_q} T_q^* h\|^2 + \|D_{\hat{T}_p} h\|^2 \\ &= \|D_{\hat{T}_q} h\|^2 + \|D_{\hat{T}_p} T_p^* h\|^2. \end{aligned}$$

This implies that the map

$$U : \{D_{\hat{T}_q} T_q^* h, D_{\hat{T}_p} h : h \in \mathcal{H}\} \to \{D_{\hat{T}_q} h, D_{\hat{T}_p} T_p^* h : h \in \mathcal{H}\},$$

defined by

$$(D_{\hat{T}_q} T_q^* h, D_{\hat{T}_p} h) \mapsto (D_{\hat{T}_q} h, D_{\hat{T}_p} T_p^* h),$$

is a well-defined isometry. By adding, if necessary, an infinite dimensional Hilbert space $\mathcal{D}$, we extend U to a unitary map, again denoted by U, from $\mathcal{D} \oplus \mathcal{D}_{\hat{T}_q} \oplus \mathcal{D}_{\hat{T}_p}$ onto itself. Then, setting

$$\mathcal{E} = \mathcal{D} \oplus \mathcal{D}_{\hat{T}_q} \oplus \mathcal{D}_{\hat{T}_p},$$

we have a unitary map $U \in \mathcal{B}(\mathcal{E})$ such that

$$U(0_{\mathcal{D}}, D_{\hat{T}_q} T_q^* h, D_{\hat{T}_p} h) = (0_{\mathcal{D}}, D_{\hat{T}_q} h, D_{\hat{T}_p} T_p^* h),$$

for all $h \in \mathcal{H}$. The equality

$$\|D_{\hat{T}_{pq}} h\|^2 = \|D_{\hat{T}_q} h\|^2 + \|D_{\hat{T}_p} T_p^* h\|^2,$$

again implies that the map $V : \mathcal{D}_{\hat{T}_{pq}} \to \mathcal{E}$ defined by

$$V(D_{\hat{T}_{pq}} h) = (0_{\mathcal{D}}, D_{\hat{T}_q} h, D_{\hat{T}_p} T_p^* h),$$

for $h \in \mathcal{H}$, is an isometry. Now by Lemma 5.1, it follows that $\hat{T}_{pq} \in \mathbb{S}_{n-1}(\mathcal{H})$ is a pure tuple. Consider the canonical isometric map $\Pi_{pq} : \mathcal{H} \to H^2_{\mathcal{D}_{\hat{T}_{pq}}}(\mathbb{D}^{n-1})$ for $\hat{T}_{pq}$ such that (5.1) holds. Then as in (5.2), set

$$\Pi_{V,pq} = (I_{H^2(\mathbb{D}^{n-1})} \otimes V) \circ \Pi_{pq} \in \mathcal{B}(\mathcal{H}, H^2_{\mathcal{E}}(\mathbb{D}^{n-1})).$$

Therefore, the isometry $\Pi_{V,pq}$ dilates $\hat{T}_{pq}$ on $\mathcal{H}$ to $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_{n-1}} \otimes I_{\mathcal{E}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n-1})$. We now prove that

$$\Pi_{V,pq} T_p^* = M_{\Phi_p}^* \Pi_{V,pq},$$

and

$$\Pi_{V,pq} T_q^* = M_{\Phi_q}^* \Pi_{V,pq},$$

for some inner polynomials $\Phi_p, \Phi_q \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^{n-1})$ in z_p variable and of degree at most one and

$$\Phi_p(\boldsymbol{z})\Phi_q(\boldsymbol{z}) = \Phi_q(\boldsymbol{z})\Phi_p(\boldsymbol{z}) = z_p I_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$. To this end, let $\iota_p : \mathcal{D}_{\hat{T}_p} \hookrightarrow \mathcal{E}$ and $\iota_q : \mathcal{D} \oplus \mathcal{D}_{\hat{T}_q} \hookrightarrow \mathcal{E}$ be the inclusion maps defined by

$$\iota_p(h_p) = (0, 0, h_p),$$

and

$$\iota_q(h, h_q) = (h, h_q, 0),$$

for all $h_p \in \mathcal{D}_{\hat{T}_p}, h_q \in \mathcal{D}_{\hat{T}_q}$ and $h \in \mathcal{D}$. Let P_p be the orthogonal projection of $\mathcal{E}$ onto $\mathcal{D}_{\hat{T}_p}$. Since

$$\begin{bmatrix} P_p & \iota_q \\ \iota_q^* & 0 \end{bmatrix} : \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{\hat{T}_q}) \to \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{\hat{T}_q}),$$

is a unitary, it follows that

$$U_1 = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_p & \iota_q \\ \iota_q^* & 0 \end{bmatrix},$$

is a unitary operator on $\mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{\hat{T}_q})$. Clearly

$$U_1 = \begin{bmatrix} UP_p & U\iota_q \\ \iota_q^* & 0 \end{bmatrix}.$$

We now prove that the unitary U_1 satisfies the condition of Lemma 5.2. Let $h \in \mathcal{H}$. Then

$$\begin{aligned}
U_1(VD_{\hat{T}_{pq}}h, 0_{\mathcal{D}}, D_{\hat{T}_q}T_p^*T_q^*h) &= U_1(0_{\mathcal{D}}, D_{\hat{T}_q}h, D_{\hat{T}_p}T_p^*h, 0_{\mathcal{D}}, D_{\hat{T}_q}T_p^*T_q^*h) \\
&= (U(0_{\mathcal{D}}, D_{\hat{T}_q}T_p^*T_q^*h, D_{\hat{T}_p}T_p^*h), 0_{\mathcal{D}}, D_{\hat{T}_q}h) \\
&= (0_{\mathcal{D}}, D_{\hat{T}_q}T_p^*h, D_{\hat{T}_p}T_p^{*2}h, 0_{\mathcal{D}}, D_{\hat{T}_q}h) \\
&= (VD_{\hat{T}_{pq}}T_p^*h, 0_{\mathcal{D}}, D_{\hat{T}_q}h).
\end{aligned}$$

Similarly, if we consider the unitary

$$U_2 = \begin{bmatrix} P_p^\perp & \iota_p \\ \iota_p^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix},$$

on $\mathcal{E} \oplus \mathcal{D}_{\hat{T}_p}$, then, again using the fact that

$$U_2 = \begin{bmatrix} P_p^\perp U^* & \iota_p \\ \iota_p^* U^* & 0 \end{bmatrix},$$

it follows that

$$U_2(VD_{\hat{T}_{pq}}h, D_{\hat{T}_p}T_p^*T_q^*h) = (VD_{\hat{T}_{pq}}T_q^*h, D_{\hat{T}_p}h),$$

for all $h \in \mathcal{H}$. Therefore by Lemma 5.2, we have $\Pi_{V,pq}T_p^* = M_{\Phi_p}^*\Pi_{V,pq}$ and $\Pi_{V,pq}T_q^* = M_{\Phi_q}^*\Pi_{V,pq}$, where

$$\Phi_p(\boldsymbol{z}) = (P_p + z_p P_p^\perp)U^*,$$

and

$$\Phi_q(\boldsymbol{z}) = U(P_p^\perp + z_p P_p),$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$, are the transfer functions corresponding to the unitaries U_1^* and U_2^* respectively. Also we have

$$\Phi_p(\boldsymbol{z})\Phi_q(\boldsymbol{z}) = \Phi_q(\boldsymbol{z})\Phi_p(\boldsymbol{z}) = z_p I_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$. This completes the proof of the theorem. □

Some remarks on the above dilation result are now in order.

Remark 1: For the base case $n = 3$, a closely related result to Theorem 5.3 was obtained in [14] as follows: Let $(T_1, T_2, T_3) \in \mathcal{T}^3(\mathcal{H})$, and let $T_3 = T_1T_2$ be a pure contraction. Then (T_1, T_2, T_3) on $\mathcal{H}$ dilates to $(M_{\Phi_1}, M_{\Phi_2}, M_z)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ where $\mathcal{E}$ is a Hilbert space, $\Phi_1, \Phi_2 \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ are inner polynomials of degree ≤ 1, and

$$\Phi_1(z)\Phi_2(z) = \Phi_2(z)\Phi_1(z) = zI_{\mathcal{E}},$$

for all $z \in \mathbb{D}$. Here (M_{Φ_1}, M_{Φ_2}) is a Berger, Coburn and Lebow pair of commuting isometries [8]. Our approach to Theorem 5.3 is partially motivated by the above result. More specifically, in Theorem 5.3, the isometric pair (M_{Φ_p}, M_{Φ_q}) is a one variable (in z_p) Berger, Coburn and Lebow pair of commuting isometries on $H^2_{\mathcal{E}}(\mathbb{D}^{n-1})$ in the following sense:

$$\Phi_p(\boldsymbol{z})\Phi_q(\boldsymbol{z}) = \Phi_q(\boldsymbol{z})\Phi_p(\boldsymbol{z}) = z_p I_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$.

Remark 2: Let $\mathcal{E}$ be a Hilbert space, and let $(M_{\varphi_1}, M_{\varphi_2})$ be a Berger, Coburn and Lebow pair of commuting isometries on $H^2_{\mathcal{E}}(\mathbb{D})$, that is, φ_1 and φ_2 be two inner functions in $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ and

$$\varphi_1(z)\varphi_2(z) = \varphi_2(z)\varphi_1(z) = zI_{\mathcal{E}},$$

for all $z \in \mathbb{D}$. For $1 \leq p < q \leq n$, define $\Phi_p(\boldsymbol{z}) = \varphi_1(z_p)$ and $\Phi_q(\boldsymbol{z}) = \varphi_2(z_p)$, $\boldsymbol{z} \in \mathbb{D}^{n-1}$. Then Φ_p and Φ_q in $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^{n-1})$ are inner polynomials in z_p of degree at most one, and

$$\Phi_p(\boldsymbol{z})\Phi_q(\boldsymbol{z}) = \Phi_q(\boldsymbol{z})\Phi_p(\boldsymbol{z}) = z_pI_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \mathbb{D}^{n-1}$. Let $\mathcal{Q}$ be a joint invariant subspace for

$$(M^*_{z_1}, \ldots, M^*_{z_{p-1}}, M^*_{\Phi_p}, M^*_{z_{p+1}}, \ldots, M^*_{z_{q-1}}, M^*_{\Phi_q}, M^*_{z_q}, \ldots, M^*_{z_{n-1}}),$$

and let

$$T_i = \begin{cases} P_{\mathcal{Q}}M_{z_i}|_{\mathcal{Q}} & \text{if } 1 \leq i < q, i \neq p, \\ P_{\mathcal{Q}}M_{\Phi_i}|_{\mathcal{Q}} & \text{if } i = p, q, \\ P_{\mathcal{Q}}M_{z_{i-1}}|_{\mathcal{Q}} & \text{if } q < i \leq n. \end{cases}$$

It is then easy to see that $(T_1, \ldots, T_n) \in \mathcal{T}^n_{p,q}(\mathcal{Q})$. Therefore

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_{p+1}}, \ldots, M_{z_{q-1}}, M_{\Phi_q}, M_{z_q}, \ldots, M_{z_{n-1}}),$$

is the model n-tuple of isometries for n-tuples of commuting contractions in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

Remark 3: The previous remark gives a list of non-trivial examples of n-tuples of operators in $\mathcal{T}^n_{p,q}(\mathcal{H})$. Also observe that if $T \in \mathcal{T}^n(\mathcal{H})$ is doubly commuting, that is, $T_i^*T_j = T_jT_i^*$ for all $1 \leq i < j \leq n$, then

$$\mathbb{S}^{-1}_{n-1}(\hat{T}_p, \hat{T}_p^*) = \prod_{i \neq p}(I_{\mathcal{H}} - T_iT_i^*),$$

for all $p \in \{1, \ldots, n\}$. Hence, if $T \in \mathcal{T}^n(\mathcal{H})$ is a doubly commuting pure tuple, then $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$ for any $1 \leq p < q \leq n$. We refer to [18] for examples of n-tuples of operators in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

We conclude by recording the von Neumann inequality for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$. The proof follows easily, as pointed out earlier (see the introduction), from the dilation result, Theorem 5.3.

Theorem 5.4. *If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, then for all $p \in \mathbb{C}[z_1, \ldots, z_n]$, the following holds:*

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\boldsymbol{z} \in \mathbb{D}^n} |p(\boldsymbol{z})|.$$

Note that the above von Neumann inequality generalizes the one considered by Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman [18] to a large class of tuples in $\mathcal{T}^n(\mathcal{H})$ (see Subsection 2.3).

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